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Nambu–Dirac manifolds

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Abstract

We present a generalization of the Dirac structure in the direction of the Nambu–Poisson structure. It is shown that with every integrable Nambu–Dirac structure there is associated a Leibniz algebroid, which yields a singular foliation endowed with a closed form. The examples of Nambu–Dirac manifolds are Dirac manifolds, Nambu–Poisson manifolds and manifolds with closed forms. A different Leibniz algebroid structure associated with a Nambu–Poisson structure is adopted for this generalization.

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1. Introduction

A Dirac structure is the notion introduced by Courant and Weinstein [4] and studied by Courant [2], in order to give a unified description of the geometry based on Hamiltonian vector fields. The name 'Dirac' derives from the fact that it is a generalization of the Dirac bracket discovered by Dirac in his theory of constraints, which is an induced Poisson bracket on a second-class constrained submanifold of a phase space endowed with a singular Lagrangian function. A Dirac structure on a manifold P is defined as a subbundle of $TP \oplus T^*P$ which is maximally isotropic with respect to the bilinear form $\langle (X, \omega), (Y, \mu)\rangle_+ = \frac{1}{2}(\omega(Y) + \mu(X))$ and whose sections are closed under the skew-symmetric bracket

$$[(X, \omega), (Y, \mu)] = ([X, Y],\ \mathcal{L}_X\mu - \mathcal{L}_Y\omega + d(\omega(Y))).$$

The basic examples of Dirac manifolds are Poisson manifolds and presymplectic manifolds (i.e. manifolds endowed with closed two-forms). With the above bracket, a Dirac structure is naturally a Lie algebroid, whose anchor map is the projection to the first component. As a result, it gives a singular foliation of P (in the sense of Stefan and Sussman). Each leaf of the foliation has a induced presymplectic form. On a Poisson manifold, the induced foliation and form is just the leafwise symplectic foliation, and the Lie algebroid structure is isomorphic to that on the cotangent bundle induced from the Poisson structure. It has been shown that the level surface $Q = J^{-1}(\mu)$ of an equivariant momentum map J on a Poisson manifold P has a Dirac structure whose leaves are the intersection of Q with leaves of P.

The Nambu–Poisson structure was introduced by Takhtajan [18] to formalize the bracket proposed by Nambu [16] and was investigated by many authors [6,8,14,15]. A Nambu–Poisson manifold of order p is a manifold endowed with a p-vector fields satisfying

$$\mathcal{L}_{X_{f_1\cdots f_{p-1}}}\Pi = 0$$

where $X_{f_1,\ldots,f_{p-1}} = \Pi(df_1, \ldots, df_{p-1})$ is a Hamiltonian vector field for functions $f_1, \ldots, f_{p-1}$. An equivalent condition is given for all $(p-1)$-forms in section 2. The Poisson structure is considered as the Nambu–Poisson structure of order 2. Recently, in [10] the authors proved that the bundle of $(p-1)$-forms on a Nambu–Poisson manifold (P, Π) has an induced Leibniz algebroid structure, a non-commutative variation of the Lie algebroid, with an anchor map $\Pi : \bigwedge^{p-1} T^*P \to TP$ and the bracket

$$[\![\alpha, \beta]\!] = \mathcal{L}_{\Pi(\alpha)}\beta + (-1)^p\,(\Pi(d\alpha))\,\beta;$$

the notion of Leibniz algebra was introduced by Loday [13] as a vector space endowed with a bilinear bracket satisfying the Leibniz identity

$$[\![a_1, [\![a_2, a_3]\!]]\!] = [\![[\![a_1, a_2]\!], a_3]\!] + [\![a_2, [\![a_1, a_3]\!]]\!].$$

A Leibniz algebroid is defined as a vector bundle with certain additional conditions as in the case of a Lie algebroid. Although this formally corresponds to the fact that the cotangent bundle of a Poisson manifold has a Lie algebroid structure, it does not holds generally for $p = 2$.

In this paper, we extend the Dirac structure to higher order for a unified description of the geometry based on a Leibniz algebroid, and elucidate its geometrical properties. The key to extension is an alternative Leibniz algebroid structure on the bundle of $(p-1)$-forms on a Nambu–Poisson manifold, to which we refer through the Nambu–Dirac framework in section 4. A Nambu–Dirac structure of order p is defined as a subbundle of $TP \oplus \bigwedge^{p-1} T^*P$ which satisfies some isotropy and integrability which agrees the definition of Dirac structure for $p = 2$. It is shown that a Nambu–Dirac structure of order p forms a Leibniz algebroid and yields a singular foliation endowed with a closed p-form. On a certain kind of Nambu–Dirac manifold, functions transversal to the kernel of this form have a Nambu–Poisson bracket. Considering that the Dirac structure has been dealt with in papers [3, 5, 11, 12], it would be interesting to investigate the Nambu–Dirac structure further.

2. Preliminaries

First, we review Nambu–Poisson manifolds, Leibniz algebroids and Dirac manifolds. We also give an equivalent definition of Nambu–Poisson structure which is applicable to all forms, not only exterior products of exact 1-forms.

2.1. Nambu–Poisson manifolds

Let P be an n-dimensional smooth manifold. A Nambu–Poisson bracket of order p ($p \leqslant n$) on P is a p-linear skew-symmetric map $\{\ldots\} : C^\infty(P) \times \cdots \times C^\infty(P) \to C^\infty(P)$ satisfying

(i) (Leibniz rule)

$$\{f_1, \ldots, f_{p-1}, g_1g_2\} = \{f_1, \ldots, f_{p-1}, g_1\}g_2 + g_1\{f_1, \ldots, f_{p-1}, g_2\}$$

(ii) (Fundamental identity)

$$\{f_1, \ldots, f_{p-1}, \{g_1, \ldots, g_p\}\} = \sum_{i=1}^{p} \{g_1, \ldots, \{f_1, \ldots, f_{p-1}, g_i\}, \ldots, g_p\}$$

for $f_1, \ldots, f_{p-1}, g_1, \ldots, g_p \in C^\infty(P)$.

It follows that a Nambu–Poisson bracket $\{\ldots\}$ is equivalently defined by a p-vector Π satisfying

$$\mathcal{L}_{X_{f_1 \cdots f_{p-1}}} \Pi = 0 \tag{1}$$

where $\Pi(df_1, \ldots, df_p) = \{f_1, \ldots, f_p\}$ for $f_1, \ldots, f_p \in C^\infty(P)$. The pair (P, Π) is called a Nambu–Poisson manifold of order p, and Π is called a Nambu–Poisson structure on P. Poisson manifolds are just Nambu–Poisson manifolds of order 2.

A point x of a Nambu–Poisson manifold (P, Π) is said to be regular if $\Pi(x) \neq 0$. The following theorem states that a Nambu–Poisson structure of order $p \geqslant 3$ is isomorphic to the standard structure locally around a regular point.

Theorem 2.1 ([6,8,15]). *Let P be an n-dimensional smooth manifold and Π a p-vector, $p \geqslant 3$. Then Π defines a Nambu–Poisson structure on P if and only if for any regular point $x \in P$, there exists a system of local coordinates $(x_1, \ldots, x_n)$ such that*

$$\Pi = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_p}.$$

On a Nambu–Poisson manifold (P, Π) of order $p \geqslant 3$, the characteristic distribution $\mathcal{D}$ given by $\mathcal{D}(x) = \Pi(\bigwedge^{p-1} T_x^* P)$ is completely integrable, thus it defines a singular foliation in the sense of Sussmann [17]. At a regular point x, the leaf through x is a p-dimensional manifold and the induced structure comes from a volume form. At a point where Π vanishes, the leaf is the point itself and the induced structure is trivial.

The following theorem gives an equivalent condition which is applicable to all forms, not only exterior products of exact one-forms.

Theorem 2.2. *A p-vector Π becomes a Nambu–Poisson structure of order p if*

$$\mathcal{L}_{\Pi(\alpha)} \Pi = (-1)^p (\Pi(d\alpha))\, \Pi \tag{2}$$

for any $(p-1)$-form α. Conversely, a Nambu–Poisson structure of order p which we assume to be decomposable when $p = 2$ satisfies the condition above.

Proof. Suppose that Π satisfies the condition (1). At a regular point, since Π is decomposable, we have

$$\begin{aligned} \mathcal{L}_{\Pi(g\alpha)} \Pi &= \mathcal{L}_{gX_{f_1 \cdots f_{p-1}}} \Pi \\ &= g\mathcal{L}_{X_{f_1 \cdots f_{p-1}}} \Pi - X_{f_1 \ldots f_{p-1}} \wedge \Pi(dg) \\ &= (-1)^p (\Pi(dg \wedge df_1 \wedge \cdots \wedge df_{p-1}))\, \Pi \\ &= (-1)^p (\Pi(d(g\alpha)))\, \Pi \end{aligned}$$

for any $\alpha = df_1 \wedge \cdots \wedge df_{p-1}$ and function g. At a point where $\Pi = 0$, by a computation similar to that above, we have $\mathcal{L}_{\Pi(g\alpha)} \Pi = g\mathcal{L}_{X_{f_1 \cdots f_{p-1}}} \Pi - X_{f_1 \ldots f_{p-1}} \wedge \Pi(dg) = 0$ for any $\alpha = df_1 \wedge \cdots \wedge df_{p-1}$ and g, and simultaneously $(-1)^p(\Pi(d\alpha))\, \Pi = 0$. □

Corollary 2.3. *A p-vector Π becomes a Nambu–Poisson structure of order p if and only if*

$$\mathcal{L}_{\Pi(\alpha)} \Pi = (-1)^p \left((\Pi(d\alpha))\, \Pi - \frac{1}{p} (\Pi \wedge \Pi)(d\alpha) \right)$$

for any $(p-1)$-form α.

Proof. For $p = 2$, this holds since $2[\Pi(\alpha), \Pi] = 2(\Pi(d\alpha))\,\Pi - (\Pi \wedge \Pi)(d\alpha)$. For $p \geqslant 3$, it follows from the decomposability of Π. □

The condition (2) is more easily dealt with than (1). For example:

Proposition 2.4. *A p-vector Π satisfying (2) is decomposable around a regular point.*

Proof. We do this by induction: suppose $p = 2$ and

$$\Pi = \sum_{i<j} c_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

locally. We may assume $c_{12} \neq 0$. Since

$$\begin{aligned} \mathcal{L}_{\Pi(f\alpha)}\Pi &= f\mathcal{L}_{\Pi(\alpha)}\Pi - \Pi(\alpha) \wedge \Pi(df) \\ &= (-1)^p f(\Pi(d\alpha))\,\Pi - \Pi(\alpha) \wedge \Pi(df) \end{aligned} \tag{3}$$

for any $(p-1)$-form α and function f, we have

$$(-1)^p c_{12}\,\Pi = \mathcal{L}_{\Pi(x_1 dx_2)}\Pi = \Pi(dx_1) \wedge \Pi(dx_2).$$

Now consider the case $p \geqslant 3$. We may take a function f and a closed $(p-1)$-form α such that $\Pi(df \wedge \alpha) \neq 0$. Then, again by (3), we have

$$\begin{aligned} (-1)^p \Pi(df \wedge \alpha)\,\Pi &= \mathcal{L}_{\Pi(f\alpha)}\Pi \\ &= -\Pi(\alpha) \wedge \Pi(df). \end{aligned}$$

It is easily checked that the $(p-1)$-vector $\Pi(df)$ satisfies (2), thus Π is decomposable. □

2.2. *Leibniz algebroids*

A Leibniz algebra is a vector space V endowed with a bilinear map $[\![,]\!] : V \times V \to V$ satisfying the Leibniz identity

$$[\![a_1, [\![a_2, a_3]\!]]\!] = [\![[\![a_1, a_2]\!], a_3]\!] + [\![a_2, [\![a_1, a_3]\!]]\!]$$

for $a_1, a_2, a_3 \in V$. The map $[\![,]\!]$ is called the Leibniz bracket or the Leibniz algebra structure on V. Note that if $[\![,]\!]$ is skew symmetric, then the Leibniz identity is the Jacobi identity and $(V, [\![,]\!])$ is a Lie algebra.

The Leibniz algebroid is defined in the same way by generalizing the notion of the Lie algebroid.

Definition 2.5 ([10]). *The Leibniz algebroid is a smooth vector bundle $\Pi : A \to P$ with a Leibniz algebra structure $[\![,]\!]$ on $\Gamma(A)$ and a bundle map $\rho : A \to TP$, called an anchor, such that the induced map $\rho : \Gamma(A) \to \Gamma(TP)$ satisfies the following properties:*

(i) (Leibniz algebra homomorphism)

$$\rho([\![x, y]\!]) = [\rho(x), \rho(y)]$$

(ii) (derivation law)

$$[\![x, fy]\!] = ((\rho(x))f)\,y + f[\![x, y]\!]$$

for all $x, y \in \Gamma(A)$ and $f \in C^\infty(P)$.

As in the case of a Lie algebroid, a Leibniz algebroid generates a singular foliation.

Theorem 2.6. *Let $(A, [\![,]\!], \rho)$ be a Leibniz algebroid over P. Then $\rho(A)$ is a integrable distribution.*

Proof. Let $e_1, \ldots, e_n$ be the local basis of $\Gamma(A)$. Then we have

$$[\![e_i, e_j]\!] = \sum_{k=1}^{n} c_{ij}^k e_k$$

for some $c_{ij}^k \in C^\infty(P)$. Since $\rho(e_1), \ldots, \rho(e_n)$ is a local generator of $\rho(A)$, it follows that

$$[\rho(e_i), \rho(e_j)] = \rho([\![e_i, e_j]\!]) = \sum_{k=1}^{n} c_{ij}^k \rho(e_k).$$

Therefore, $\rho(A)$ satisfies the integrability condition of Sussmann [17]. □

For any Poisson manifold (P, π), the cotangent bundle T^*P has the Lie algebroid structure whose anchor is the Poisson bundle map $\pi : T^*P \to TP$ and whose bracket is given by

$$[\alpha, \beta] = \mathcal{L}_{\pi(\alpha)}\beta - \mathcal{L}_{\pi(\beta)}\alpha - d(\pi(\alpha, \beta)) \tag{4}$$

(see [19]). Similarly, a Nambu–Poisson manifold of order $p \geqslant 3$ has an associated Leibniz algebroid.

Theorem 2.7 ([10]). *Let (P, Π) be a Nambu–Poisson manifold of order p for which we assume Π is decomposable when $p = 2$. Then the triple $(\bigwedge^{p-1} T^*P, [\![,]\!], \Pi)$ is a Leibniz algebroid over P, where $\Pi : \bigwedge^{p-1} T^*P \to TP$ and the bracket $[\![,]\!]$ is defined by*

$$[\![\alpha, \beta]\!] = \mathcal{L}_{\Pi(\alpha)}\beta + (-1)^p\,(\Pi(d\alpha))\,\beta \tag{5}$$

for $\alpha, \beta \in \Gamma(\bigwedge^{p-1} T^*P)$.

Proof. As this is proved in [10], we give a computational proof using (2). It is easy to see that $\Pi([\![\alpha, \beta]\!]) = [\Pi(\alpha), \Pi(\beta)]$ and $[\![\alpha, f\beta]\!] = ((\Pi(\alpha))f)\,\beta + f\,[\![\alpha, \beta]\!]$. Now we prove the Leibniz identity. We calculate

$$\begin{aligned}
[\![\alpha, [\![\beta, \gamma]\!]]\!] &= \mathcal{L}_{\Pi(\alpha)}(\mathcal{L}_{\Pi(\beta)}\gamma + (-1)^p(\Pi(d\beta))\,\gamma) \\
&\quad + (-1)^p(\Pi(d\alpha))\,(\mathcal{L}_{\Pi(\beta)}\gamma + (-1)^p(\Pi(d\beta))\,\gamma) \\
&= \mathcal{L}_{\Pi(\alpha)}\mathcal{L}_{\Pi(\beta)}\gamma + (-1)^p(\,\mathcal{L}_{\Pi(\alpha)}(\Pi(d\beta))\,)\,\gamma + (-1)^p(\Pi(d\beta))\,\mathcal{L}_{\Pi(\alpha)}\gamma \\
&\quad + (-1)^p(\Pi(d\alpha))\,\mathcal{L}_{\Pi(\beta)}\gamma + (\Pi(d\alpha))(\Pi(d\beta))\,\gamma \\
[\![[\![\alpha, \beta]\!], \gamma]\!] &= \mathcal{L}_{[\Pi(\alpha),\Pi(\beta)]}\gamma \\
&\quad + (-1)^p(\Pi(\mathcal{L}_{\Pi(\alpha)}d\beta))\,\gamma + \Pi(d(\Pi(d\alpha)) \wedge \beta)\,\gamma + (\Pi(d\alpha))(\Pi(d\beta))\gamma \\
&= \mathcal{L}_{[\Pi(\alpha),\Pi(\beta)]}\gamma \\
&\quad + (-1)^p(\Pi(\mathcal{L}_{\Pi(\alpha)}d\beta))\,\gamma - (-1)^p(\mathcal{L}_{\Pi(\beta)}(\Pi(d\alpha)))\,\gamma + (\Pi(d\alpha))(\Pi(d\beta))\gamma \\
&= \mathcal{L}_{[\Pi(\alpha),\Pi(\beta)]}\gamma + (-1)^p(\,\mathcal{L}_{\Pi(\alpha)}(\Pi(d\beta))\,)\,\gamma - (-1)^p(\mathcal{L}_{\Pi(\beta)}(\Pi(d\alpha)))\,\gamma \\
[\![\beta, [\![\alpha, \gamma]\!]]\!] &= \mathcal{L}_{\Pi(\beta)}\mathcal{L}_{\Pi(\alpha)}\gamma + (-1)^p(\,\mathcal{L}_{\Pi(\beta)}(\Pi(d\alpha))\,)\,\gamma + (-1)^p(\Pi(d\alpha))\,\mathcal{L}_{\Pi(\beta)}\gamma \\
&\quad + (-1)^p(\Pi(d\beta))\,\mathcal{L}_{\Pi(\alpha)}\gamma + (\Pi(d\beta))(\Pi(d\alpha))\,\gamma.
\end{aligned}$$

Thus we have $[\![\alpha, [\![\beta, \gamma]\!]]\!] = [\![[\![\alpha, \beta]\!], \gamma]\!] + [\![\beta, [\![\alpha, \gamma]\!]]\!]$. □

Note that (5) does not holds generally for $p = 2$, that is, Poisson manifolds. In fact, for decomposable Poisson structures, (5) equals (4).

The converse of the theorem above also holds. Indeed, since the anchor is a Leibniz algebra homomorphism, we deduce that Π satisfies the condition (2).

2.3. Dirac manifolds

We mainly summarize basic definitions and properties of Dirac manifolds investigated by Courant [2].

Let V be an n-dimensional vector space. We may define the nondegenerate bilinear forms called the plus pairing on $V \oplus V^*$ by

$$\langle (X, \omega), (Y, \mu) \rangle_+ = \frac{1}{2}(\omega(Y) + \mu(X))$$

where $(X, \omega), (Y, \mu) \in V \oplus V^*$. A Dirac structure on V is a subspace L of $V \oplus V^*$ which is maximally isotropic under the plus pairing $\langle , \rangle_+$. From the definition, it follows that $\dim L = n$.

Definition 2.8. *Let P be a smooth manifold. A smooth subbundle $L \subset TP \oplus T^*P$ is called an almost Dirac structure on P if the fibre of L is maximally isotropic under the plus pairing $\langle , \rangle_+$ at each point.*

We denote the projection from $TP \oplus T^*P$ onto the first component by ρ, and that onto the second component by ρ'. Moreover, we consider $L \cap TP$ as a subset of either L or TP as the case may be; similarly for $L \cap T^*P$. In this notation, we have

$$\ker \rho|_L = L \cap T^*P$$
$$\ker \rho'|_L = L \cap TP.$$

We also consider TP and T^*P as subbundles of $TP \oplus T^*P$ as the case may be.

An almost Dirac structure has the following properties:

Proposition 2.9.

$$\mathrm{Ann}\, \rho(L) = L \cap T^*P$$
$$\mathrm{Ann}\, \rho'(L) = L \cap TP.$$

Here Ann $\rho(L)$ *is the annihilator of $\rho(L)$ with respect to the natural pairing between TP and T^*P.*

These are called the characteristic equations of L.

Proposition 2.10. *An almost Dirac structure L on P induces a linear map $\Omega : \rho(L) \wedge \rho(L) \to \mathbb{R}$ defined by*

$$\Omega(X, Y) = \omega(Y) = -\mu(X)$$

for any $(X, \omega), (Y, \mu) \in L$.

The integrability condition of Dirac structures is given as follows: L is called a Dirac structure on P and (P, L) is said to be a Dirac manifold when $\Gamma(L)$ is closed under the skew-symmetric bracket

$$[e, e'] = ([X, Y],\ \mathcal{L}_X\mu - \mathcal{L}_Y\omega + d(\omega(Y))) \tag{6}$$

where $e = (X, \omega)$, $e' = (Y, \mu)$. The three-tensor T_L on L defined by

$$T_L(e \otimes e' \otimes e'') = \langle [e, e'], e'' \rangle_+$$

vanishes if and only if L is integrable.

Proposition 2.11. *L is integrable if and only if $(L, [\ ,\ ], \rho|_L)$ is a Lie algebroid.*

Therefore, if L is integrable then $\rho(L)$ generates a singular foliation. Moreover, by proposition 2.10, there is a two-form Ω on each leaf of the foliation. A direct computation shows that it is closed on each leaf. Thus we have:

Theorem 2.12. *A Dirac manifold has a singular foliation by presymplectic leaves.*

Example 2.13.

(i) Let (P, Ω) be a presymplectic manifold, that is, Ω is a closed two-form on P. Then the graph of $\Omega : TP \to T^*P$ is a Dirac structure on P. It yields the foliation by one leaf which is P itself, and the two-form of proposition 2.9 is Ω.

(ii) Let (P, π) be a Poisson manifold. Then the graph of $\pi : T^*P \to TP$ is a Dirac structure on P. The induced two-form is dual to π on leaves, that is, we obtain the leafwise symplectic foliation. The associated Lie algebroid structure on the cotangent bundle is isomorphic to that on the Dirac structure.

(iii) Consider the singular Poisson structure on $\mathbb{R}^3$ given by

$$\{x, y\} = \frac{1}{z}, \qquad \{x, z\} = 0, \qquad \{y, z\} = 0$$

which appears in the problem of guiding centre motion in the plane. We may rewrite this Poisson structure as the Dirac structure on $\mathbb{R}^3$ spanned by

$$\left(-\frac{\partial}{\partial x}, z\,dy\right), \qquad \left(\frac{\partial}{\partial y}, z\,dx\right), \qquad (0, dz)$$

which has no singularity. Its leaves are the planes $z =$ constant and the induced two-forms are given by $\Omega = z\,dx \wedge dy$.

(iv) Let (P, Λ, E) be a Jacobi manifold, that is, Λ be a bivector field and E be a vector field such that $[\Lambda, \Lambda] = 2E \wedge \Lambda$ and $[E, \Lambda] = 0$. Denote by X_u the Hamiltonian vector field of $u \in C^\infty(P)$, that is, $X_u = \Lambda(du) + uE$. Around a point where $E \neq 0$, a Jacobi structure is locally a Dirac structure spanned by

$$(X_{u_1}, du_1), \ldots, (X_{u_{n-1}}, du_{n-1}), (E, 0)$$

where $u_1, \ldots, u_{n-1} \in C^\infty(P)$ such that $du_1, \ldots, du_{n-1}$ spans Ann E.

The maximality for the isotropy is not necessarily needed to yield a presymplectic foliation; we shall discuss this in the framework of Nambu–Dirac structures in the next section.

The Dirac manifold is suitable for the description of a dynamical system with constraints. It is known that an almost symplectic manifold (P, Ω) endowed with a regular foliation such that the pull-back of Ω to each leaf is symplectic has an induced Poisson structure, whose bracket is called the Dirac bracket. The following theorem is its generalization.

Theorem 2.14. *Let P be a manifold endowed with a two-form Ω and a regular foliation S such that the pull-back Ω' of Ω to each leaf is closed on it. Then S induces a Dirac structure on P.*

Proof. Let

$$L = \{(X, \omega) \in TP \oplus T^*P \mid X \in S \text{ and } \omega|_S = \Omega'(X)\}.$$

It follows that $\langle L, L\rangle_+ = 0$ and

$$\begin{aligned} \dim L_x &= \dim \rho\,(L)_x + \dim\ker \rho\,(L)_x \\ &= \dim S_x + \dim(L \cap T^*P)_x \\ &= \dim S_x + \dim \operatorname{Ann} S_x \\ &= n \end{aligned}$$

for any $x \in P$. The smoothness of L follows from the regularity of S. The integrability holds since Ω is closed. □

We remark that, as in the case of Poisson manifolds, a singular foliation endowed with a closed two-form does not always yield a Dirac structure.

Now let us consider a dynamical system with constraints on some phase space. Generally, a submanifold of a symplectic manifold is presymplectic. Thus if considered constraints on the phase space give a regular foliation, which always holds locally, then it induces a Dirac structure on the phase space by the theorem above, and it determines equations of motion under the constraints. We obtain the Dirac bracket when constraints are all second class. See [7] for how presymplectic structures on a constraint submanifold describe the Dirac theory of constraints.

3. Nambu–Dirac manifolds

3.1. Almost Nambu–Dirac structures

Let P be an n-dimensional smooth manifold. We put $\mathcal{T}_1 = TP \oplus \bigwedge^{p-1} T^*P$, $\mathcal{T}_{p-1} = \bigwedge^{p-1} TP \oplus T^*P$ where $2 \leqslant p \leqslant n$ and $N = \dim \mathcal{T}_1(x) = \dim \mathcal{T}_{p-1}(x)$ where $x \in P$. We have two natural pairings between $\mathcal{T}_1$ and $\mathcal{T}_{p-1}$, namely, bilinear maps from $\mathcal{T}_1 \times \mathcal{T}_{p-1}$ to $\mathbb{R}$ defined by

$$\langle (X, \boldsymbol{\omega}), (\boldsymbol{Y}, \mu)\rangle_+ = \tfrac{1}{2}(\boldsymbol{\omega}(\boldsymbol{Y}) + (-1)^p \mu(X))$$
$$\langle (X, \boldsymbol{\omega}), (\boldsymbol{Y}, \mu)\rangle_- = \tfrac{1}{2}(\boldsymbol{\omega}(\boldsymbol{Y}) - (-1)^p \mu(X))$$

for all $(X, \boldsymbol{\omega}) \in \mathcal{T}_1$, $(\boldsymbol{Y}, \mu) \in \mathcal{T}_{p-1}$. Let us denote both projections from $\mathcal{T}_1, \mathcal{T}_{p-1}$ onto the first components by the same letter ρ, and those onto the second components by ρ'.

Definition 3.1. *A smooth subbundle $L \subset \mathcal{T}_1$ is said to be an almost Nambu–Dirac structure of order p on P if*

$$(\boldsymbol{\omega}(X') + \boldsymbol{\omega}'(X))|_{\bigwedge^{p-2} \rho(L)} = 0 \tag{7}$$

for any $(X, \boldsymbol{\omega}), (X', \boldsymbol{\omega}') \in L$ and

$$\bigwedge^{p-1} \rho(L) = \rho(L^\perp) \tag{8}$$

where $L^\perp$ denotes the annihilator of L with respect to the plus pairing $\langle , \rangle_+$.

The condition (7) is equivalent to the requirement that $(X \wedge \boldsymbol{Z}, \boldsymbol{\omega}(\boldsymbol{Z})) \in L^\perp$ for any $(X, \boldsymbol{\omega}) \in L$ and $\boldsymbol{Z} \in \bigwedge^{p-2} \rho(L)$. Note that this implies $\bigwedge^{p-1} \rho(L) \subset \rho(L^\perp)$. We also remark $\dim L(x) + \dim L^\perp(x) = N$ where $x \in P$.

By definition, it follows that

$$\ker \rho|_L = L \cap \bigwedge^{p-1} T^*P \qquad \ker \rho|_{L^\perp} = L^\perp \cap T^*P$$
$$\ker \rho'|_L = L \cap TP \qquad \ker \rho'|_{L^\perp} = L^\perp \cap \bigwedge^{p-1} TP.$$

Note that we consider $L \cap TP$ as a subset of either L or TP as the case may be; similarly for $L \cap \bigwedge^{p-1} T^*P$, $L^\perp \cap T^*P$ and $L^\perp \cap \bigwedge^{p-1} TP$. We also consider $\mathcal{T}_1, \mathcal{T}_{p-1}$ as subbundles of $\mathcal{T}_1 \oplus \mathcal{T}_{p-1}$ as the case may be.

Example 3.2.

(i) Let P be a manifold and Ω a closed p-form on P. Consider the graph of $\Omega : TP \to \bigwedge^{p-1} T^*P$

$$L = \{(X, \Omega(X)) \in \mathcal{T}_1 \mid X \in TP\}.$$

Then we have

$$L^{\perp} = \left\{ (\boldsymbol{Y}, \Omega(\boldsymbol{Y})) \in \mathcal{T}_{p-1} | \boldsymbol{Y} \in \bigwedge^{p-1} TP \right\}$$

and it follows that L is an almost Nambu–Dirac structure of order p. Conversely, if an almost Nambu–Dirac structure on P is given by the graph of a map $A : TP \to \bigwedge^{p-1} T^*P$, then a p-form Ω is defined by $\Omega(X \wedge \boldsymbol{Y}) = A(X, \boldsymbol{Y})$.

(ii) Let (P, Π) be a Nambu–Poisson manifold of order $p \geqslant 3$. Consider the graph of Π

$$L = \left\{ (\Pi(\omega), \omega) \in \mathcal{T}_1 | \omega \in \bigwedge^{p-1} T^*P \right\}.$$

We see that L is an almost Nambu–Dirac structure of order p. Indeed, let $X_1, \ldots, X_n$ be the vector fields which span TP and $\Pi = X_1 \wedge \cdots X_p$ locally. Then we deduce $\boldsymbol{\omega}(\Pi(\boldsymbol{\omega}') \wedge \boldsymbol{X}_{ij}) + \boldsymbol{\omega}'(\Pi(\boldsymbol{\omega}) \wedge \boldsymbol{X}_{ij}) = 0$ for any $\boldsymbol{X}_{ij} = X_1 \wedge \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge X_p$. As $\rho(L)$ is spanned by $X_1, \ldots, X_p$, we have (7) of definition 3.1. Furthermore, since

$$L^{\perp} = \left\{ (\Pi(\mu), \mu) \in \mathcal{T}_{p-1} \mid \mu \in T^*P \right\}$$

and Π is decomposable, the condition (8) of definition 3.1 holds.

Conversely, if an almost Nambu–Dirac structure on P is given by the graph of a map $B : \bigwedge^{p-1} T^*P \to TP$, a p-vector Π is defined by $\Pi(\boldsymbol{\omega}, \mu) = B(\boldsymbol{\omega}, \mu)$. Moreover, since $L^{\perp}$ is equal to the graph of $(-1)^{p-1} B^*$, by (8) we obtain rank $B = p$, that is, Π is decomposable.

(iii) Any Dirac structure is an almost Nambu–Dirac structure of order 2. Conversely, suppose that (P, L) is an almost Nambu–Dirac structure of order 2. The condition (7) is equivalent to the requirement that L is isotropic under $\langle , \rangle_+$. Assume $L \subset L^{\perp}$. For any $\omega \in L^{\perp} \cap T^*P$ we have

$$\langle (0, \omega), L^{\perp} \rangle_+ = \langle \omega | \rho(L^{\perp}) \rangle = \langle \omega | \rho(L) \rangle = \langle L, (0, \omega) \rangle_+ = 0$$

thus we have $L^{\perp} \cap T^*P \subset L \cap T^*P$. This and $L \subset L^{\perp}$ imply that $L \cap T^*P = L^{\perp} \cap T^*P$. Therefore,

$$\begin{aligned} \dim L &= \dim \rho(L) + \dim \ker \rho|_L \\ &= \dim \rho(L^{\perp}) + \dim \ker \rho|_{L^{\perp}} \\ &= \dim L^{\perp} \end{aligned}$$

and it follows that $\dim L = n$. Thus L is a Dirac structure on P.

In the above examples, there are the integrability conditions, namely, the Jacobi identity, the fundamental identity and the vanishing of the integrability tensor. We shall consider the integrability condition of an almost Nambu–Dirac structure later.

Proposition 2.9 is generalized as follows:

Proposition 3.3. *The following equations hold.*

$$\operatorname{Ann} \rho(L^{\perp}) = L \cap \bigwedge^{p-1} T^*P \qquad \operatorname{Ann} \rho(L) = L^{\perp} \cap T^*P \tag{9}$$

$$\operatorname{Ann} \rho'(L^{\perp}) = L \cap TP \qquad \operatorname{Ann} \rho'(L) = L^{\perp} \cap \bigwedge^{p-1} TP. \tag{10}$$

Proof. We prove (10). Since $\langle \rho'(L^\perp) \,|\, L \cap TP \rangle = \langle L \cap TP, L^\perp \rangle_+ = 0$, we have $\rho'(L^\perp) \subset \mathrm{Ann}(L \cap TP)$. Similarly, we have $\rho'(L) \subset \mathrm{Ann}\big(L^\perp \cap \bigwedge^{p-1} T^*P\big)$. Thus at each point it follows that

$$\begin{aligned}
\dim L - \dim\ker \rho'|_L &= \dim \rho'(L) \\
&\leqslant \dim \mathrm{Ann}\Big(L^\perp \cap \bigwedge^{p-1} T^*P\Big) \\
&= \dim \bigwedge^{p-1} T^*P - \dim\ker \rho'|_{L^\perp} \\
\dim L^\perp - \dim\ker \rho'|_{L^\perp} &= \dim \rho'(L^\perp) \\
&\leqslant \dim \mathrm{Ann}\,\big(L \cap T^*P\big) \\
&= \dim T^*P - \dim\ker \rho'|_L .
\end{aligned}$$

The sum of the leftmost sides of the two inequalities is equal to that of their rightmost sides of them. Thus we have the equations (10). In the same way, we obtain the equations (9). □

We call (9) and (10) the characteristic equations of L. Since these equations hold, we can obtain a linear map $\Omega : \bigwedge^p \rho(L) \to \mathbb{R}$ as follows: define $\tilde{\Omega} : \rho(L) \to \rho(L^\perp)^*$ and $\tilde{\tilde{\Omega}} : \rho(L^\perp) \to \rho(L)^*$ by $\tilde{\Omega}(\rho(e_1)) = \rho'(e_1)|_{\rho(L^\perp)}$ and $\tilde{\tilde{\Omega}}(\rho(e_{p-1})) = \rho'(e_{p-1})|_{\rho(L)}$ respectively; if $e_1, e_1' \in L$ satisfies $\rho(e_1) = \rho(e_1')$, then $e_1 - e_1' \in \ker \rho|_L = \mathrm{Ann}\,\rho(L^\perp)$, and accordingly $\rho'(e_1 - e_1')|_{\rho(L^\perp)} = 0$, which means that $\tilde{\Omega}$ is well defined. We may consider $\tilde{\Omega} : \rho(L) \times \rho(L^\perp) \to \mathbb{R}$. Similarly for $\tilde{\tilde{\Omega}}$. Clearly it holds that $\tilde{\Omega}(X, \boldsymbol{Y}) = (-1)^{p-1}\tilde{\tilde{\Omega}}(\boldsymbol{Y}, X)$ for $X \in \rho(L), \boldsymbol{Y} \in \rho(L^\perp)$. Furthermore,

$$\tilde{\Omega}(X, X' \wedge \boldsymbol{Z}) = \boldsymbol{\omega}(X' \wedge \boldsymbol{Z}) = -\boldsymbol{\omega}'(X \wedge \boldsymbol{Z}) = -\tilde{\tilde{\Omega}}(X', X \wedge \boldsymbol{Z})$$

for any $(X, \boldsymbol{\omega}), (X', \boldsymbol{\omega}') \in L$ and $\boldsymbol{Z} \in \bigwedge^{p-2} \rho(L)$. Hence we obtain $\Omega : \bigwedge^p \rho(L) \to \mathbb{R}$ by $\Omega(X \wedge \boldsymbol{Y}) = \tilde{\Omega}(X, \boldsymbol{Y})$.

Thus we obtain the following:

Proposition 3.4. *An almost Nambu–Dirac structure L induces a linear map* $\Omega : \bigwedge^p \rho(L) \to \mathbb{R}$.

In fact, in the whole discussion above, the condition (8) is not necessarily needed. Indeed, since $\bigwedge^{p-1} \rho(L) \subset \rho(L^\perp)$, we obtain $\Omega : \bigwedge^p \rho(L) \to \mathbb{R}$ by restricting $\tilde{\Omega}$ to $\bigwedge^p \rho(L)$. We call such a structure L an almost generalized Nambu–Dirac structure. Every isotropic subbundle of $TP \oplus T^*P$ is an almost generalized Nambu–Dirac structure of order 2, and vice versa.

The following proposition does not hold for generalized Nambu–Dirac structures in general.

Proposition 3.5. *Let L be an almost Nambu–Dirac structure of order p and consider*

$$\begin{aligned}
&\ker \Omega = \{X \in \rho(L) \,|\, \Omega(X) = 0\} \\
&\ker_{p-1} \Omega = \Big\{\boldsymbol{Y} \in \bigwedge^{p-1} \rho(L) \,\Big|\, \Omega(\boldsymbol{Y}) = 0\Big\}.
\end{aligned}$$

Then it follows that $\ker \Omega = L \cap TP$ *and* $\ker_{p-1} \Omega = L^\perp \cap \bigwedge^{p-1} TP$.

Proof. Since $\rho'(L \cap TP) = 0$, we have $L \cap TP \subset \ker \Omega$. Conversely, take $e_1 \in L$ such that $\Omega(\rho(e_1)) = 0$. Since $\rho(e_1)|_{\rho'(L^\perp)} = \rho'(e_1)|_{\rho(L^\perp)} = 0$, we have $\ker \Omega \subset \mathrm{Ann}\,\rho'(L^\perp) = L \cap TP$. Thus we have $\ker \Omega = L \cap TP$. In the same way, we have $\ker_{p-1} \Omega = L^\perp \cap \bigwedge^{p-1} TP$. □

Example 3.6. Let L be a generalized Nambu–Dirac structure of order 3 on $\mathbb{R}^3$ spanned at each point by

$$\left(\frac{\partial}{\partial x_1}, dx_2 \wedge dx_3\right)$$

where (x_1, x_2, x_3) are the standard coordinates. Since $\rho(L)$ is spanned by $\partial/\partial x_1$ and the induced structure $\Omega = 0$, we deduce $\ker \Omega \neq 0$, while $L \cap TP = 0$. Similarly, since $L^\perp$ is spanned by

$$\left(\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}, dx_1\right), \quad \left(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}, 0\right), \quad \left(\frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1}, 0\right), \quad (0, dx_2), \quad (0, dx_3)$$

we also obtain $\ker_{p-1} \Omega \neq L^\perp \cap \bigwedge^{p-1} TP$.

However, generalized Nambu–Dirac structures also have some of the geometric properties which Nambu–Dirac structures do.

3.2. Integrability of Nambu–Dirac structures

We define a bilinear bracket on sections of $\mathcal{T}_1$ by

$$[\![e, e']\!] = ([X, Y], \mathcal{L}_X \boldsymbol{\mu} - \iota_Y d\boldsymbol{\omega}) \tag{11}$$

for $e = (X, \boldsymbol{\omega}), e' = (Y, \boldsymbol{\mu}) \in \Gamma(\mathcal{T}_1)$.

Definition 3.7. *Let $L \subset \mathcal{T}_1$ be an almost Nambu–Dirac structure of order p. L is called a Nambu–Dirac structure, or an integrable Nambu–Dirac bundle, if $\Gamma(L)$ is closed under the bracket $[\![,]\!]$. We call (P, L) a Nambu–Dirac manifold of order p.*

Note that $\mathcal{L}_X\boldsymbol{\mu} - \iota_Y d\boldsymbol{\omega} = \mathcal{L}_X\boldsymbol{\mu} - \mathcal{L}_Y\boldsymbol{\omega} + d(\boldsymbol{\omega}(Y))$, thus this is a natural generalization of (6). As in the Dirac case, we shall introduce the integrability tensor of L.

Definition 3.8. *Let $L \subset \mathcal{T}_1$ be an almost Nambu–Dirac structure of order p. We define $T_L : L \otimes L \otimes L^\perp \to \mathbb{R}$ by*

$$T_L(e, e', e'') = \langle [\![e, e']\!], e'' \rangle_+$$

where $e, e' \in \Gamma(L)$ and $e'' \in \Gamma(L^\perp)$.

We shall check that T_L is a tensor. For any $e = (X, \boldsymbol{\omega}), e' = (Y, \boldsymbol{\mu}) \in \Gamma(L), e'' \in \Gamma(L^\perp)$ and $f \in C^\infty(P)$, it follows that

$$\begin{aligned}
\langle [\![e, fe']\!], e'' \rangle_+ &= \langle ([X, fY], \mathcal{L}_X(f\boldsymbol{\mu}) - \iota_{fY} d\boldsymbol{\omega}), e'' \rangle_+ \\
&= \langle ((Xf)Y + f[X, Y], (Xf)\boldsymbol{\mu} + f\mathcal{L}_X\boldsymbol{\mu} - f\iota_Y d\boldsymbol{\omega}), e'' \rangle_+ \\
&= \langle ((Xf)Y, (Xf)\boldsymbol{\mu}), e'' \rangle_+ + f \langle [\![e, e']\!], e'' \rangle_+ \\
&= f \langle [\![e, e']\!], e'' \rangle_+ \\
\langle [\![fe, e']\!], e'' \rangle_+ &= \langle ([fX, Y], \mathcal{L}_{fX}\boldsymbol{\mu} - \iota_Y d(f\boldsymbol{\omega})), e'' \rangle_+ \\
&= f \langle [\![e, e']\!], e'' \rangle_+ \\
&\quad + \langle (-(Yf)X, df \wedge \boldsymbol{\mu}(X) - (Yf)\boldsymbol{\omega} + df \wedge \boldsymbol{\omega}(Y)), e'' \rangle_+ \\
&= f \langle [\![e, e']\!], e'' \rangle_+ - \langle (Yf)e, e'' \rangle_+ + \langle (0, df \wedge (\boldsymbol{\mu}(X) + \boldsymbol{\omega}(Y))), e'' \rangle_+ \\
&= f \langle [\![e, e']\!], e'' \rangle_+.
\end{aligned}$$

Therefore,

$$T_L(e, fe', e'') = T_L(e, e', fe'') = fT_L(e, e', e'').$$

Obviously $T_L(e, e', fe'') = fT_L(e, e', e'')$, thus T_L is a three-tensor.

In the computation above we have shown:

Proposition 3.9. *For* $e = (X, \boldsymbol{\omega}), e' = (Y, \boldsymbol{\mu}) \in \Gamma(L)$ *and* $f \in C^\infty(P)$,

$$[\![e, fe']\!] = f\,[\![e, e']\!] + ((\rho(e))f)\,e'$$
$$[\![fe, e']\!] = f\,[\![e, e']\!] - (Yf)\,e + (0,\ df \wedge (\boldsymbol{\mu}(X) + \boldsymbol{\omega}(Y))).$$

We call T_L the integrability tensor of L since:

Proposition 3.10. *L is a Nambu–Dirac structure if and only if* $T_L = 0$.

Proof. This follows from the definition of T_L. □

Now we shall see that a Nambu–Dirac structure has an associated Leibniz algebroid, as a Dirac structure has an associated Lie algebroid.

Theorem 3.11. *An almost Nambu–Dirac structure L is integrable if and only if $(L, [\![,]\!], \rho|_L)$ is a Leibniz algebroid.*

Proof. As we have already shown in proposition 3.9 that the bracket $[\![,]\!]$ has the derivation property, the assertion holds only if L satisfies the Leibniz identity. Since

$$\begin{aligned}
\rho'([\![e, [\![e', e'']\!]]\!]) &= \mathcal{L}_X(\mathcal{L}_Y\boldsymbol{\zeta} - \iota_Z\, d\boldsymbol{\mu}) - \iota_{[Y,Z]}\, d\boldsymbol{\omega} \\
&= \mathcal{L}_X\mathcal{L}_Y\boldsymbol{\zeta} - (\mathcal{L}_X\, d\boldsymbol{\mu})(Z) - d\boldsymbol{\mu}(\mathcal{L}_X Z) - d\boldsymbol{\omega}(\mathcal{L}_Y Z) \\
\rho'([\![[\![e, e']\!], e'']\!]) &= \mathcal{L}_{[X,Y]}\boldsymbol{\zeta} - \iota_Z\, d(\mathcal{L}_X\boldsymbol{\mu} - \iota_Y\, d\boldsymbol{\omega}) \\
&= \mathcal{L}_{[X,Y]}\boldsymbol{\zeta} - (\mathcal{L}_X\, d\boldsymbol{\mu})(Z) + (\mathcal{L}_Y\, d\boldsymbol{\omega})(Z) \\
\rho'([\![e', [\![e, e'']\!]]\!]) &= \mathcal{L}_Y\mathcal{L}_X\boldsymbol{\zeta} - (\mathcal{L}_Y\, d\boldsymbol{\omega})(Z) - d\boldsymbol{\omega}(\mathcal{L}_Y Z) - d\boldsymbol{\mu}(\mathcal{L}_X Z)
\end{aligned}$$

for $e = (X, \boldsymbol{\omega}), e' = (Y, \boldsymbol{\mu}), e'' = (Z, \boldsymbol{\zeta}) \in \Gamma(L)$, we have

$$\rho'([\![e, [\![e', e'']\!]]\!]) = \rho'([\![[\![e, e']\!], e'']\!]) + \rho'([\![e', [\![e, e'']\!]]\!]).$$

It is obvious that

$$\rho([\![e, [\![e', e'']\!]]\!]) = \rho([\![[\![e, e']\!], e'']\!]) + \rho([\![e', [\![e, e'']\!]]\!]).$$

Therefore, the Leibniz identity holds. □

By theorem 2.6, we have a singular foliation of a Nambu–Dirac manifold (P, L). In addition, the leaves are endowed with a p-form Ω by proposition 3.4. Let us examine this induced structure on leaves.

Lemma 3.12. *Let* $\iota_L : L \oplus L^\perp \to \mathcal{T}_1 \oplus \mathcal{T}_{p-1}$ *be the inclusion and* $\rho_L : L \oplus L^\perp \to TP \oplus \bigwedge^{p-1} TP$ *be the projection. Then it follows that* $\rho_L^*\tilde{\Omega} = \iota_L^*(\langle , \rangle_-)$.

Proof. $\rho_L^*\tilde{\Omega}(e_1, e_{p-1}) = \tilde{\Omega}(X, \boldsymbol{Y}) = \langle e_1, e_{p-1}\rangle_- = \iota_L^*(\langle e_1, e_{p-1}\rangle_-)$ for any $e_1 = (X, \omega) \in L, e_1' = (\boldsymbol{Y}, \mu) \in L^\perp$. □

This lemma ensures that Ω is smooth. Now we compute $d\Omega$. Suppose that $e_1 = (X, \omega), e_1' = (Y, \boldsymbol{\mu}) \in L$ and $e_{p-1} = (\boldsymbol{Z}, \zeta) \in L^\perp$ so that $\boldsymbol{Z} \in \bigwedge^{p-1} \rho(L)$. Considering $\mathcal{T}_1, \mathcal{T}_{p-1} \subset \mathcal{T}_1 \oplus \mathcal{T}_{p-1}$, we have

$$\begin{aligned}
\rho_L^* d\Omega(e_1, e_1', e_{p-1}) &= d\Omega(X, Y, \boldsymbol{Z}) \\
&= (\iota_X\, d\Omega)(Y, \boldsymbol{Z}) \\
&= (\mathcal{L}_X\Omega)(Y, \boldsymbol{Z}) - d\boldsymbol{\omega}(Y, \boldsymbol{Z}) \\
&= X(\Omega(Y, \boldsymbol{Z})) - \Omega(\mathcal{L}_X Y, \boldsymbol{Z}) - \Omega(Y, \mathcal{L}_X \boldsymbol{Z}) - d\boldsymbol{\omega}(Y, \boldsymbol{Z}) \\
&= 2\langle [\![e_1, e_1']\!], e_{p-1}\rangle_+ .
\end{aligned} \tag{12}$$

Therefore, $d\Omega$ is closed if and only if T_L vanishes. Thus we have shown:

Theorem 3.13. *A Nambu–Dirac manifold has a singular foliation whose leaves are endowed with a closed p-form.*

Note that on a leaf whose dimension is less than p, the induced structure is automatically trivial. We also remark that (integrable) generalized Nambu–Dirac structures are defined in the same way and they also have foliations endowed with closed forms.

As theorem 2.14, we have the following:

Theorem 3.14. *Let P be a manifold endowed with a p-form Ω and a regular foliation S such that the pull-back Ω' of Ω to each leaf is closed on it. Then S induces a Nambu–Dirac structure on P.*

Proof. Let

$$L = \{(X, \boldsymbol{\omega}) \in \mathcal{T}_1 \mid X \in S \text{ and } \boldsymbol{\omega}|_S = \Omega'(X)\}.$$

As in the Dirac case, it is proved that L is a Nambu–Dirac structure on P. □

In fact, if and only if L is integrable, $L \oplus L^\perp$ has a Leibniz algebroid structure whose anchor is the projection to TP and whose bracket is

$$\{\boldsymbol{e}, \boldsymbol{e}'\} = (([X, Y],\ \mathcal{L}_X \boldsymbol{\mu} - \iota_Y\, d\boldsymbol{\omega}),\ ([X, \boldsymbol{Y}],\ \mathcal{L}_X \mu - \iota_{\boldsymbol{Y}}\, d\boldsymbol{\omega}))$$

for any $\boldsymbol{e} = ((X, \boldsymbol{\omega}), (\boldsymbol{X}, \omega)),\ \boldsymbol{e}' = ((Y, \boldsymbol{\mu}), (\boldsymbol{Y}, \mu)) \in L \oplus L^\perp$. Indeed, the derivation law and the Leibniz identity hold as well as L. The closedness under the bracket holds because under the notation $\boldsymbol{e} = (e_1, e_{p-1}) \in L \oplus L^\perp$ it follows that $\{\boldsymbol{e}, \boldsymbol{e}'\}_1 = [\![e_1, e_1']\!]$ and, by the equation (12),

$$\begin{aligned}
2\langle [\![e_1, e_1']\!], e''_{p-1}\rangle_+ &= d\Omega(X, Y, \boldsymbol{Z})\\
&= (-1)^p\, (\mathcal{L}_Y \Omega)(\boldsymbol{Z}, X) - (-1)^p\, d\boldsymbol{\mu}(\boldsymbol{Z}, X)\\
&= (-1)^p\, Y(\Omega(\boldsymbol{Z}, X)) - (-1)^p\, \Omega(\boldsymbol{Z}, \mathcal{L}_Y X) + \Omega(X, \mathcal{L}_Y \boldsymbol{Z})\\
&\quad -(-1)^p\, d\boldsymbol{\mu}(\boldsymbol{Z}, X)\\
&= (-1)^p\, Y(\zeta(X)) - (-1)^p\, \zeta(\mathcal{L}_Y X) + \boldsymbol{\omega}(\mathcal{L}_Y \boldsymbol{Z}) - (-1)^p\, (\iota_{\boldsymbol{Z}}\, d\boldsymbol{\mu})(X)\\
&= (-1)^p\, (\mathcal{L}_Y \zeta - \iota_{\boldsymbol{Z}}\, d\boldsymbol{\mu})(X) + \boldsymbol{\omega}(\mathcal{L}_Y \boldsymbol{Z})\\
&= 2\langle e_1, \{e', e''\}_{p-1}\rangle_+
\end{aligned}$$

for any $\boldsymbol{e} = ((X, \boldsymbol{\omega}), (\boldsymbol{X}, \omega)),\ \boldsymbol{e}' = ((Y, \boldsymbol{\mu}), (\boldsymbol{Y}, \mu)),\ \boldsymbol{e}'' = ((Z, \boldsymbol{\zeta}), (\boldsymbol{Z}, \zeta)) \in L \oplus L^\perp$. Note that this induces the same foliation and structure on leaves as L does.

3.3. Infinitesimal Nambu–Dirac automorphisms

Let (P, L) be a Nambu–Dirac manifold of order p and $(X, \boldsymbol{\omega}), (Y, \boldsymbol{\mu}) \in \Gamma(L)$. We define $\mathcal{L}_X(Y, \boldsymbol{\mu}) = (\mathcal{L}_X Y, \mathcal{L}_X \boldsymbol{\mu})$. Since

$$\begin{aligned}
[\![(X, \boldsymbol{\omega}), (Y, \boldsymbol{\mu})]\!] &= ([X, Y], \mathcal{L}_X \boldsymbol{\mu} - \iota_Y\, d\boldsymbol{\omega})\\
&= \mathcal{L}_X(Y, \boldsymbol{\mu}) - (0, \iota_Y\, d\boldsymbol{\omega})
\end{aligned}$$

we have

$$\begin{aligned}
\langle \mathcal{L}_X(Y, \boldsymbol{\mu}), (\boldsymbol{Z}, \zeta)\rangle_+ &= \langle [\![(X, \boldsymbol{\omega}), (Y, \boldsymbol{\mu})]\!], (\boldsymbol{Z}, \zeta)\rangle_+ + \langle (0, \iota_Y\, d\boldsymbol{\omega}), (\boldsymbol{Z}, \zeta)\rangle_+\\
&= \tfrac{1}{2}\, d\boldsymbol{\omega}(Y, \boldsymbol{Z})
\end{aligned}$$

for any $(\boldsymbol{Z}, \zeta) \in \Gamma(L^\perp)$. Therefore, $\mathcal{L}_X L \subset L$ if and only if $d\boldsymbol{\omega}|_{\bigwedge^p \rho(L)} = 0$.

Now we consider the invariance of the induced structure Ω on each leaf under flows.

Theorem 3.15. *If $(X, \boldsymbol{\omega})$ is a section of L, we have $\mathcal{L}_X \Omega = d\boldsymbol{\omega}|_{\bigwedge^p \rho(L)}$.*

Proof. Since Ω is closed on each leaf, we have $\mathcal{L}_X \Omega = d\iota_X \Omega$. □

Thus we deduce that $\mathcal{L}_X L \subset L$ if and only if $\mathcal{L}_X \Omega = 0$.

3.4. The admissible functions

A function f on a Dirac manifold (P, L) is said to be admissible if df is a section of $\rho'(L)$. The set of admissible functions $\mathrm{Adm}(P)$ naturally forms a Poisson algebra (see [2]).

Now we discuss the functions on a Nambu–Dirac manifold (P, L) of order p. We call L a strong Nambu–Dirac structure when it holds that

$$(\boldsymbol{X}(\mu) + \boldsymbol{Y}(\omega))|_{\bigwedge^{p-2}\rho'(L^\perp)} = 0 \tag{13}$$

for any $(\boldsymbol{X}, \omega), (\boldsymbol{Y}, \mu) \in L^\perp$. Clearly, as well as (7) this implies $\bigwedge^{p-1}\rho'(L^\perp) \subset \rho'(L)$. Dirac manifolds, Nambu–Poisson manifolds, multisymplectic manifolds of dimensional order and manifolds endowed with decomposable closed p-forms are all strong Nambu–Dirac manifolds. Note that (13) is independent of (8) for $p \geqslant 3$.

Example 3.16 (Generalized strong Nambu–Dirac structure). Let (x_1, x_2, x_3, x_4) be standard coordinates on $\mathbb{R}^4$. The subbundle $L \subset T\mathbb{R}^4 \oplus \bigwedge^3 T^*\mathbb{R}^4$ spanned at each point by

$$\left(\frac{\partial}{\partial x_1}, 0\right), (0, dx_2 \wedge dx_3 \wedge dx_4), (0, dx_1 \wedge dx_3 \wedge dx_4), (0, dx_1 \wedge dx_2 \wedge dx_4)$$

is a generalized strong Nambu–Dirac structure of order 4. Indeed, $L^\perp$ is spanned by

$$\left(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}, 0\right), \qquad (0, dx_2), \qquad (0, dx_3), \qquad (0, dx_4).$$

A function f on a strong Nambu–Dirac manifold (P, L) is said to be admissible if df is a section of $\rho'(L^\perp)$. We denote by $\mathrm{Adm}(P)$ the set of admissible functions on P. By an argument similar to that in proposition 3.4, we have a linear map $\Pi : \bigwedge^p \rho'(L) \to \mathbb{R}$. Thus we may define a skew p-bracket on $\mathrm{Adm}(P)$ by

$$\{f_1, \ldots, f_p\} = \Pi(df_1 \wedge \cdots \wedge df_p).$$

For $f_1, \ldots, f_p \in \mathrm{Adm}(P)$, there exists a vector field $X_{f_1 \ldots f_{p-1}}$ and a $(p-1)$-vector field $\boldsymbol{X}_{f_p}$ such that $e_{f_1 \ldots f_{p-1}} = (X_{f_1 \ldots f_{p-1}}, df_1 \wedge \cdots \wedge df_{p-1})$ is a section of L and $e_{f_p} = (\boldsymbol{X}_{f_p}, df_p)$ is a section of $L^\perp$. We remark that $X_{f_1 \ldots f_{p-1}}$ and $\boldsymbol{X}_{f_p}$ are not uniquely determined in general.

Lemma 3.17. $\{\ldots\}$ *satisfies the derivation law.*

Proof. For any $g, h \in \mathrm{Adm}(P)$, take $\boldsymbol{X}_g, \boldsymbol{X}_h \in \rho(\Gamma(L^\perp))$ and define $\boldsymbol{X}_{gh} = g\boldsymbol{X}_h + h\boldsymbol{X}_g$. Then we have $(\boldsymbol{X}_{gh}, d(gh)) = ge_h + he_g \in \Gamma(L^\perp)$, which implies that gh is admissible. It is obvious that

$$\{f_1, \ldots, f_{p-1}, gh\} = \{f_1, \ldots, f_{p-1}, g\}h + g\{f_1, \ldots, f_{p-1}, h\}$$

for any $f_1, \ldots, f_{p-1} \in \mathrm{Adm}(P)$. □

Lemma 3.18. $\{\ldots\}$ *satisfies the fundamental identity.*

Proof. Let $f_1, \ldots, f_{p-1}, g_1, \ldots, g_{p-1}, h$ be admissible functions on P and take $e_{f_1 \ldots f_{p-1}}$, $e_{g_1 \ldots g_{p-1}} \in \Gamma(L)$. Since

$$[\![e_{f_1 \ldots f_{p-1}}, e_{g_1 \ldots g_{p-1}}]\!] = \left([X_{f_1 \ldots f_{p-1}}, X_{g_1 \ldots g_{p-1}}], \mathcal{L}_{X_{f_1 \cdots f_{p-1}}}(dg_1 \wedge \cdots \wedge dg_{p-1})\right)$$

and

$$\mathcal{L}_{X_{f_1 \cdots f_{p-1}}}(dg_1 \wedge \cdots \wedge dg_{p-1}) = \sum_{i=1}^{p} dg_1 \wedge \cdots \wedge d\{f_1, \ldots, f_{p-1}, g_i\} \wedge \cdots \wedge dg_{p-1}$$

we have

$$
\begin{aligned}
0 &= \langle [\![e_{f_1 \dots f_{p-1}}, e_{g_1 \dots g_{p-1}}]\!], e_h \rangle_+ \\
&= \left(\sum_{i=1}^{p} dg_1 \wedge \cdots \wedge d\{f_1, \dots, f_{p-1}, g_i\} \wedge \cdots \wedge dg_{p-1} \right)(\boldsymbol{X}_h) \\
&\quad + (-1)^p \left[X_{f_1 \dots f_{p-1}}, X_{g_1 \dots g_{p-1}} \right] h \\
&= (-1)^{p-1} \sum_{i=1}^{p-1} \{g_1, \dots, \{f_1, \dots, f_{p-1}, g_i\}, \dots, g_{p-1}, h\} \\
&\quad + (-1)^p \left[X_{f_1 \dots f_{p-1}}, X_{g_1 \dots g_{p-1}} \right] h.
\end{aligned}
$$

This implies

$$
\left[X_{f_1 \dots f_{p-1}}, X_{g_1 \dots g_{p-1}} \right] h = \sum_{i=1}^{p-1} \{g_1, \dots, \{f_1, \dots, f_{p-1}, g_i\}, \dots, g_{p-1}, h\}
$$

hence the fundamental identity holds. □

Thus we have shown:

Theorem 3.19. *The set of admissible functions* $\mathrm{Adm}(P)$ *on a strong Nambu–Dirac manifold* (P, L) *naturally has a Nambu–Poisson bracket.*

On a strong Nambu–Dirac manifold (P, L), flows along leaves which keep L invariant are given by admissible functions, that is:

Theorem 3.20. *Let* (P, L) *be a strong Nambu–Dirac manifold of order* p. *Then* L *is locally invariant under* $X_{f_1 \dots f_{p-1}}$ *where* $f_1, \dots, f_{p-1}$ *are admissible functions.*

Proof. This follows from $d(df_1 \wedge \cdots \wedge df_{p-1}) = 0$. □

Now, let us consider the characteristic distribution of the induced p-form Ω on each leaf of P, that is, $\ker \Omega = L \cap TP$.

Theorem 3.21. *Let* (P, L) *be a Nambu–Dirac manifold. If* $L \cap TP$ *is a bundle, it generates a regular foliation* Φ.

This theorem results from the following general lemma:

Lemma 3.22. *Let* M *be a manifold and* Ω *a closed* p*-form on* M. *If* $\mathrm{char}\,\Omega = \{X \in TM \mid \Omega(X) = 0\}$ *is a bundle, it is involutive.*

This is a generalization of the Dirac case. On a Dirac manifold, in addition, P/Φ inherits a Poisson structure naturally, which can be generalized to strong Nambu–Dirac manifolds:

Theorem 3.23. *Let* (P, L) *be a strong Nambu–Dirac manifold and* $L \cap TP$ *a bundle. Then* P/Φ *inherits a Nambu–Poisson bracket from* $\mathrm{Adm}(P)$.

Proof. We may regard functions on the manifold P/Φ as functions constant on Φ, that is, all $f \in C^\infty(P)$ satisfying $df|_{L \cap TP} = 0$. By the characteristic equation $\mathrm{Ann}\,\rho'(L^\perp) = L \cap TP$, such functions are the admissible functions on P. □

4. Applications

We give some examples obtained from multisymplectic, Nambu–Poisson and Nambu–Jacobi structures. We also give an alternative Leibniz algebroid structure on the bundle of $(p-1)$-forms on a Nambu–Poisson manifold.

4.1. Nambu–Dirac structures induced from multisymplectic manifolds

A multisymplectic structure of order p on a manifold P, in the sense of [1], is a closed and nondegenerate p-form, where a p-form Ω is said to be nondegenerate when $\Omega : TP \to \bigwedge^{p-1} T^*P$ is injective. For $p = 2$, we rediscover a symplectic structure. It is obvious that multisymplectic manifolds and submanifolds of them are Nambu–Dirac manifolds of which foliations consist of one leaf. For example, quaternionic Kähler manifolds and their submanifolds are Nambu–Dirac manifolds.

In [1], the authors shows that for an arbitrary manifold P, the bundle $\bigwedge^{p-1} T^*P$ of $(p-1)$-forms has the canonical multisymplectic structure of order p locally represented by

$$\Omega_{\mathrm{can}} = \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_{p-1}\leqslant n} dy_{i_1\cdots i_{p-1}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{p-1}}$$

where $(x_i, y_{i_1\ldots i_{p-1}})$ are the local coordinates of $\bigwedge^{p-1} T^*P$. Therefore, a bundle of $(p-1)$-forms on an arbitrary manifold and its subbundles are all Nambu–Dirac manifolds of order p. Furthermore, consider a bundle of $(p-1)$-forms on a multisymplectic manifold (P, Ω). The image of $\Omega : TP \to \bigwedge^{p-1} T^*P$ is endowed with a closed p-form as a Nambu–Dirac structure. Thus we deduce that the tangent bundle of a multisymplectic manifold has a Nambu–Dirac structure induced from Ω.

Finally, we remark that a Nambu–Dirac manifold (P, L) satisfying $L \cap TP = 0$ has a foliation by multisymplectic leaves. While the induced leaves of a Nambu–Poisson structure have symplectic forms or volume forms, both of which are examples of multisymplectic structure, this gives a more general kind of leafwise multisymplectic foliation.

4.2. Nambu–Poisson manifolds and associated Leibniz algebroids

Now we shall discuss the Nambu–Dirac structures of Nambu–Poisson manifolds. Let (P, Π) be a Nambu–Poisson manifold of order p and L its Nambu–Dirac structure; in example 3.2 (ii) we see that L is defined by the graph of $\Pi : \bigwedge^{p-1} T^*P \to TP$, whose integrability follows from the fundamental identity via a computation using lemma 3.18. For $p \geqslant 3$, it follows from proposition 3.3 and theorem 2.1 that

$$\begin{aligned}\dim \rho(L) &= \dim TP - \dim \mathrm{Ann}\, \rho(L)\\ &= n - \dim(L^{\perp} \cap T^*P)\\ &= n - \dim \ker \Pi\\ &= \begin{cases} p & (\Pi(x) \neq 0)\\ 0 & (\Pi(x) = 0)\end{cases}\end{aligned}$$

at each point $x \in P$. Therefore, a leaf through a regular point is a p-dimensional submanifold endowed with the induced volume form Ω, whereas at a singular point the leaf is the point itself and the induced structure is trivial. Namely, we rediscover the induced foliation of a Nambu–Poisson manifold. For $p = 2$, the graph becomes a Dirac structure and we obtain the usual leafwise symplectic foliation of a Poisson manifold.

The pull-back of the Leibniz algebroid structure on L by the isomorphism $(\rho'|_L)^{-1}$ gives $\bigwedge^{p-1} T^*P$ a Leibniz algebroid structure, whose anchor is $\Pi : \bigwedge^{p-1} T^*P \to TP$ and whose bracket is

$$[\![\alpha, \beta]\!] = \mathcal{L}_{\Pi(\alpha)}\beta - \iota_{\Pi(\beta)}\, d\alpha. \tag{14}$$

This Leibniz algebroid structure disagrees with that given by theorem 2.7 since the Leibniz algebra structures (14) and (5) are different. The former is a natural generalization of Lie algebroid structures associated with Poisson manifolds, because (4) may be written in this form. It includes the case $p = 2$ naturally, while the latter does only if the Poisson structure is decomposable. In addition to the decomposable Poisson case, the two structures coincide if Π is originates from a volume form. We also deduce that $\bigwedge^{p-1} T^*P \oplus T^*P$ has the Leibniz algebroid structure with the anchor $\rho(\boldsymbol{\omega}, \omega) = \Pi(\boldsymbol{\omega})$ and the bracket

$$[\![(\boldsymbol{\omega}, \omega), (\boldsymbol{\mu}, \mu)]\!] = (\mathcal{L}_{\Pi(\omega)}\boldsymbol{\mu} - \iota_{\Pi(\mu)}\, d\boldsymbol{\omega},\ \mathcal{L}_{\Pi(\omega)}\mu - \iota_{\Pi(\mu)}\, d\boldsymbol{\omega}).$$

A Nambu–Poisson structure with singularity can be regarded as a Nambu–Dirac structure without singularity. For example, consider a singular Nambu–Poisson structure on $P = \mathbb{R}^{p+1}$

$$\Pi = \frac{1}{x_{p+1}} \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_p}$$

which is singular at $x_{p+1} = 0$. Let L be a subbundle of $\mathcal{T}_1$ spanned by

$$\left(\frac{\partial}{\partial x_i}, (-1)^{p-i} x_{p+1}\, dx_1 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_p\right), \qquad (0, dx_1 \wedge \cdots \widehat{dx_i} \ldots \widehat{dx_j} \cdots \wedge dx_p)$$

where $1 \leqslant i, j \leqslant p$. Then $L^\perp$ is spanned by

$$\left(\frac{\partial}{\partial x_1} \wedge \cdots \frac{\hat{\partial}}{\partial x_i} \cdots \wedge \frac{\partial}{\partial x_p}, (-1)^{i-1} x_{p+1}\, dx_i\right), \qquad (0, dx_{p+1})$$

where $1 \leqslant i \leqslant p$, thus L is a Nambu–Dirac structure of order p; the integrability of L is easily checked. L is smoothly defined on all $\mathbb{R}^{p+1}$. The leaves are planes $x_{p+1} =$ constant, and the induced p-forms are given by $\Omega = x_{p+1}\, dx_1 \wedge \cdots \wedge dx_p$.

4.3. Nambu–Jacobi structures at regular points

A Nambu–Jacobi structure of order $p \geqslant 3$ on a manifold P is a pair of a p-vector and a $(p-1)$-vector (Π, E) satisfying

$$\mathcal{L}_{\Pi(d\boldsymbol{f}_{p-1})}\Pi = 0 \tag{15}$$

$$\mathcal{L}_{E(d\boldsymbol{f}_{p-2})}E = 0 \tag{16}$$

$$\mathcal{L}_{E(d\boldsymbol{f}_{p-2})}\Pi = 0 \tag{17}$$

$$\mathcal{L}_{\Pi(d\boldsymbol{f}_{p-1})}E = (-1)^{p-1}\, \Pi(d(E(d\boldsymbol{f}_{p-1}))) \tag{18}$$

for any functions $f_1, \ldots, f_{p-1}$, where $d\boldsymbol{f}_{p-1}$ represents $df_1 \wedge \cdots \wedge df_{p-1}$ [14]. We give an equivalent condition using theorem 2.2:

Proposition 4.1. *For $p \geqslant 3$ for which we assume E is decomposable when $p = 3$, a p-vector Π and a $(p-1)$-vector E define a Nambu–Jacobi structure if and only if*

$$\mathcal{L}_{\Pi(\alpha)}\Pi = (-1)^p(\Pi(d\alpha))\, \Pi \tag{19}$$

$$\mathcal{L}_{E(\beta)}E = (-1)^{p-1}(E(d\beta))\, E \tag{20}$$

$$\mathcal{L}_{E(\beta)}\Pi = (-1)^{p-1}(E(d\beta))\, \Pi \tag{21}$$

$$\mathcal{L}_{\Pi(\alpha)}E = (-1)^p(\Pi(d\alpha))\, E + (-1)^{p-1}\, \Pi(d(E(\alpha))) \tag{22}$$

for any $(p-1)$-form α and $(p-2)$-form β.

Proof. By theorem 2.2, (15) and (16) are equivalent to (19) and (20) respectively. We shall show the rest. By an argument similar to that in theorem 2.2, we only have to prove the proposition at a point where $\Pi \neq 0$. Since $\iota_\theta \Pi = E$ and $\Pi(d\theta) = 0$ for some one-form θ, we have

$$\begin{aligned}
\mathcal{L}_{E(\beta)}\Pi &= (-1)^{p-1}(\Pi(d(\theta \wedge \beta)))\,\Pi \\
&= (-1)^{p-1}(\Pi(\theta \wedge d\beta))\,\Pi \\
&= (-1)^{p-1}(E(d\beta))\,\Pi, \\
\mathcal{L}_{\Pi(\alpha)}E &= \mathcal{L}_{\Pi(\alpha)}(\Pi(\theta)) \\
&= (\mathcal{L}_{\Pi(\alpha)}\Pi)\theta + \Pi(\mathcal{L}_{\Pi(\alpha)}\theta) \\
&= (\mathcal{L}_{\Pi(\alpha)}\Pi)\theta + \Pi(d\iota_{\Pi(\alpha)}\theta + \iota_{\Pi(\alpha)}d\theta) \\
&= (-1)^{p}(\Pi(d\alpha))\,\Pi(\theta) + (-1)^{p-1}\Pi(\,d(E(\alpha))\,) + \Pi(\,d\theta(\Pi(\alpha))\,) \\
&= (-1)^{p}(\Pi(d\alpha))\,\Pi(\theta) + (-1)^{p-1}\Pi(\,d(E(\alpha))\,).
\end{aligned}$$

□

The associated foliation of a Nambu–Jacobi manifold is as follows:

Theorem 4.2 ([9]). *Let (P, Π, E) be a Nambu–Jacobi manifold of order $p \geqslant 3$. Then P has the associated foliation such that*

(i) if $\Pi(x) \neq 0$, the leaf through x is a p-dimensional manifold endowed with the Nambu–Poisson structure of order p coming from Π,

(ii) if $\Pi(x) = 0$ and $E \neq 0$, the leaf through x is a $(p-1)$-dimensional manifold,

- *(a) if $p > 3$ then the induced structure is the Nambu–Poisson structure of order $p-1$ coming from E,*
- *(b) if $p = 3$ then the induced structure is a symplectic structure coming from the Poisson structure E,*

(iii) if $\Pi(x) = 0$ and $E = 0$, the leaf is the point x and the induced structure is trivial.

Now, we shall see that locally, around a regular point, a Nambu–Jacobi structure of order $p \geqslant 3$ is actually a Nambu–Dirac structure of order $p-1$. Suppose that (P, Π, E) is a Nambu–Jacobi manifold of order $p \geqslant 3$. Let us denote the characteristic distributions of Π and E by D_Π and D_E respectively. From regularity, it follows that there exists a non-zero vector field v such that $D_\Pi = \operatorname{span} v \oplus D_E$. Let

$$L = \left(E\left(\bigwedge^{p-2} \operatorname{Ann} v\right), \bigwedge^{p-2} \operatorname{Ann} v \right) \oplus \left(0, \operatorname{span} v^* \wedge \left(\operatorname{Ann} \bigwedge^{p-3} D_E\right)\right) \oplus (\operatorname{span} v, 0).$$

Then

$$L^\perp = (\,E(\operatorname{Ann} v), \operatorname{Ann} v\,) \oplus \left(\operatorname{span} v \wedge \left(\bigwedge^{p-3} D_E\right), 0\right)$$

and it follows that L is a strong Nambu–Dirac structure of order $p-1$. Indeed, since $\dim \operatorname{Ann} D_E = n - p + 1$, we may take coordinates $x_1, \ldots, x_n$ such that

$$\Pi = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{p-1}} \wedge \frac{\partial}{\partial x_n}, \qquad E = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{p-1}} \qquad v = \frac{\partial}{\partial x_n}$$

locally. L and $L^{\perp}$ are spanned by

$$\left(X^E_{x_{i_1}\ldots x_{i_{p-2}}}, dx_{i_1}\wedge\cdots\wedge dx_{i_{p-2}}\right), \qquad (0, dx_n\wedge dx_{j_1}\wedge\cdots\wedge dx_{j_{p-3}}), \qquad (v, 0)$$

and

$$\left(X^E_{x_i}, dx_i\right) \qquad \left(\frac{\partial}{\partial x_n}\wedge\frac{\partial}{\partial x_{k_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{k_{p-3}}}, 0\right)$$

respectively, where $1 \leqslant i_1, \ldots, i_{p-2} \leqslant n-1$, $(j_1, \ldots, j_{p-3}) \nsubseteq (1, \ldots, p-1)$, $1 \leqslant i \leqslant n-1$ and $(k_1, \ldots, k_{p-3}) \subset (1, \ldots, p-1)$. The induced foliation is generated by D_Π, that is, the same foliation as that of theorem 4.2. Thus the leaf through a regular point is p-dimensional, while the induced structure Ω is a $(p-1)$-form. Furthermore, $\ker\Omega$ is spanned by v and the quotient P/Φ where Φ is the characteristic foliation of Ω has a Nambu–Poisson bracket originating from E. This is isomorphic to that of admissible functions

$$\mathrm{Adm}(P) = \{f \in C^\infty(P) \mid df \in \mathrm{Ann}\, v\}.$$

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